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Hyperasymptotic expansions of confluent hypergeometric functions

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Hyperasymptotic expansions were recently introduced by Berry and Howls, and yield refined information by expanding remainders in asymptotic expansions. This paper gives a new method for obtaining hyperasymptotic expansions for integrals representing the confluent hypergeometric U-function. At each level, the remainder is exponentially small compared with the previous remainders, and the number of new terms is increasing. Three numerical illustrations confirm these exponential improvements.

1. Introduction

We use the following integral representation for one of the confluent hypergeometric functions

$$U(a, c, z) = \frac{1}{\Gamma(a)} \int_0^\infty \mathrm{e}^{-zt} t^{a-1} (1+t)^{-b}\, \mathrm{d}t, \tag{1.1}$$

where $b = a - c + 1$ and where the constant a fulfils $\operatorname{Re} a > 0$. A truncated Taylor series of $(1+t)^{-b}$ around $t = 0$ is of the form

$$(1+t)^{-b} = \sum_{n=0}^{N_0-1} \frac{(b)_n}{n!}(-t)^n + \frac{(b)_{N_0}}{N_0!}(-t)^{N_0} T_{N_0}(t), \tag{1.2}$$

where Pochhammer's symbol $(b)_n$ is defined by $(b)_n = \Gamma(b+n)/\Gamma(b)$. By substituting this Taylor series in (1.1), we obtain the asymptotic expansion

$$U(a, c, z) = z^{-a} \sum_{n=0}^{N_0-1} (-1)^n \frac{(a)_n (b)_n}{n!\, z^n} + R_{N_0}(z). \tag{1.3}$$

For fixed N_0 and large z, $|R_{N_0}(z)|$ is of order $z^{-a-N_0}(a)_{N_0}(b)_{N_0}/N_0!$. The optimum remainder is at $N_0 = [z]$, or thereabouts, when z is large. In Olver [8: p. 532] this remainder is written as

$$R_{N_0}(z) = C_{N_0}(z) \frac{(a)_{N_0}(b)_{N_0}}{N_0!\, z^{a+N_0}}.$$

Here $C_{N_0}(z)$ is called the converging factor. This factor has the integral representation

$$C_{N_0}(z) = \frac{z^{a+N_0}}{\Gamma(a+N_0)} \int_0^\infty t^{a-1+N_0} \mathrm{e}^{-zt} T_{N_0}(t)\, \mathrm{d}t, \tag{1.4}$$

and Olver obtains an asymptotic expansion for this factor by expanding $T_{N_0}(t)$ in a Taylor series around $t=1$, where $t^{a-1+N_0}e^{-zt}$ has its peak value, approximately. This is his final expansion for the function $U(a, c, z)$.

The idea of re-expanding the remainder term with a view to enhancing numerical accuracy originated in the doctoral dissertation of Stieltjes [10]. It has been developed (mostly in a formal manner) by several writers, especially Airey [1], Miller [7], and Dingle [6]. Chapter 14 of [8] is confined to cases in which rigorous analysis can be supplied.

The purpose of the paper is to describe the continuation of this process. By introducing a remainder $R_{N_1}(z)$, we truncate the asymptotic expansion of $C_{N_0}(z)$. We choose the optimal N_1 and then we expand $R_{N_1}(z)$ in a new asymptotic expansion. This method leads to a hyperasymptotic expansion. First we give a general description of hyperasymptotic expansions.

An asymptotic series is commonly divergent. Thus, an asymptotic expansion with asymptotic parameter z,

$$F(z) \sim A_0 + A_1 + A_2 + \cdots \quad \text{as } z \to \infty, \tag{1.5}$$

has to be truncated at a certain point to obtain a reasonable approximation. Such a truncated series is of the form

$$F(z) = A_0 + A_1 + \cdots + A_{N_0-1} + R_{N_0}(z), \tag{1.6}$$

where the optimal N_0 can be obtained by minimization of the remainder R_{N_0} when z is given. Generally, N_0 is a function of z. Considering R_{N_0} as a function depending on two large parameters (z and N_0), it is expanded in a new asymptotic expansion, which is truncated at N_1. Thus the approximation is of the form

$$F(z) = A_0 + \cdots + A_{N_0-1} + B_0 + \cdots + B_{N_1-1} + R_{N_1}(z), \tag{1.7}$$

where the remainder R_{N_1} appears to be exponentially small compared with R_{N_0}, as $z \to \infty$. Usually, A_n is written in the form $A_n = a_n z^{-n}$, whereas B_n will have a different representation. Finally, we obtain

$$F(z) = A_0 + \cdots + A_{N_0-1} + B_0 + \cdots + B_{N_1-1} + C_0 + C_1 + \cdots + R_{N_n}(z), \tag{1.8}$$

where R_{N_n} appears to be exponentially small compared with $R_{N_{n-1}}$, as $z \to \infty$. Expansion (1.8) is the nth-level hyperasymptotic expansion of F.

This successively exponential reduction of remainders makes the expansion (1.8) useful as an approximation of $F(z)$ for 'medium z'.

Hyperasymptotic expansions were introduced by Berry & Howls [2]. There F is a solution of a second-order differential equation. Their expansion (1.8) is obtained by using this differential equation, and N_0 is obtained by tracing the smallest term in the expansion. At that stage, R_{N_0} is seen as the Borel sum $A_{N_0} + A_{N_0+1} + A_{N_0+2} + \cdots$. With a Borel resummation Berry & Howls obtain the new asymptotic expansion $R_{N_0} \sim B_0 + B_1 + \cdots$. This method leads to a hyperasymptotic expansion of the form (1.8). Furthermore, they need $N_0 > N_1 > N_2 > \cdots$, and they prove that the optimal expansion can be obtained with $N_0 \simeq 2N_1 \simeq 4N_2 \simeq \cdots$. Thus at each level the number of new terms is smaller than the number of new terms at the previous level, such that in their final hyperasymptotic expansion the number of terms is finite and the final remainder cannot be exponentially improved.

No Borel resummation is involved in a more recent paper of Berry & Howls [3]. In [3] they obtain the same kind of expansions for integrals with saddles. They refine the method of steepest descent. The refinement is achieved by means of an exact 'resurgence relation', expressing the original integral as its truncated saddle-point asymptotic expansion plus a remainder involving the integrals through certain 'adjacent' saddles. Iteration of the resurgence relation leads to a representation of the integral as a sum of contributions associated with 'multiple scattering paths' among the saddles. Each path gives a 'hyperseries'.

In our method we also use an integral as starting point. We use an integral representation of the confluent hypergeometric U-function which is a special case of integrals of the form

$$\int_0^\infty e^{-zt} t^{a-1} f_0(t)\, dt, \qquad \operatorname{Re} a > 0, \tag{1.9}$$

where z is the asymptotic parameter and $f_0(t)$ is an analytic function on $\operatorname{Re} t \geq 0$. Expanding $f_0(t)$ in a Taylor series at $t = 0$ leads to the expansion (1.5). In our approach, N_0 is not obtained by tracing the smallest term, but it is obtained by minimizing the remainder R_{N_0}. This is done in Section 2.

The remainder is of the form

$$\int_0^\infty e^{-zt} t^{a-1+N_0} f_1(t)\, dt, \tag{1.10}$$

where N_0 is a function of z. Now we expand $f_1(t)$ at the point where $t^{a-1+N_0} e^{-zt}$ is maximal. This gives the $B_0 + B_1 + \cdots$ part, and this is done in Section 3.

In Section 4 we show how this leads to an expansion of the form (1.8). At each level of this expansion, special polynomials are needed for the representation of the new terms. These polynomials are investigated in Section 5. We obtain some recursion relations, which make these polynomials easier to handle in the numerical illustrations given in Section 6.

In that section we compare numerically the hyperasymptotic expansion obtained with the method of [2] and the expansion obtained with our method. As explained above, the hyperasymptotic expansion (1.8) obtained in [2] is finite. Our method yields an infinite analogue of (1.8), and this leads to exponential improvements which are of higher order than the exponential improvements in [2].

In the final section we describe two other methods for obtaining first-level hyperasymptotic expansions of the confluent hypergeometric U-function. These methods are introduced in Boyd [4] and Olver [9].

First we take a and b positive, and in a remark at the end of Section 4 we indicate how to extend our method to complex a and b.

2. Zeroth-level hyperasymptotic expansions

We write $z = \rho e^{i\theta}$ and we take $t = \tau e^{-i\theta}$, where $-\pi + \delta \leq \theta \leq \pi - \delta$ with δ small and positive. The factor $(1+t)^{-b}$ in the integrand of (1.1) shows that this is the maximal t-domain. In this way we obtain an analytic continuation with respect to this z-domain for $U(a, c, z)$, with

$$U(a,c,z)=\frac{\mathrm{e}^{-\mathrm{i}a\theta}}{\Gamma(a)}\int_0^\infty \mathrm{e}^{-\rho\tau}\tau^{a-1}(1+\tau\mathrm{e}^{-\mathrm{i}\theta})^{-b}\,\mathrm{d}\tau. \tag{2.1}$$

Thus we rotated the path of integration from $(0,\infty\mathrm{e}^{\mathrm{i}\theta})$ to $(0,\infty)$. The Taylor series of $(1+\tau\mathrm{e}^{-i\theta})^{-b}$ around $\tau=0$ is

$$(1+\tau\mathrm{e}^{-\mathrm{i}\theta})^{-b}=\sum_{n=0}^{\infty}\frac{(b)_n}{n!}(-\tau\mathrm{e}^{-\mathrm{i}\theta})^n. \tag{2.2}$$

If we substitute (2.2) in (2.1), we obtain the asymptotic expansion

$$U(a,c,z)\sim z^{-a}\sum_{n=0}^{\infty}(-1)^n\frac{(a)_n(b)_n}{n!\,z^n}\quad \text{as } |z|\to\infty. \tag{2.3}$$

To obtain the zeroth-level hyperasymptotic expansion, we truncate this divergent series by introducing a minimal remainder. The minimality of the remainder is based on a criterion described below. We write

$$(1+\tau\mathrm{e}^{-\mathrm{i}\theta})^{-b}=\sum_{n=0}^{N_0-1}\frac{(b)_n}{n!}(-\tau\mathrm{e}^{-\mathrm{i}\theta})^n+\tau^{N_0}f_1(\tau), \tag{2.4}$$

and we have to calculate an optimal N_0 such that the remainder is minimal. Taylor's theorem gives for $f_1(\tau)$ the integral representation

$$f_1(\tau)=\frac{1}{2\pi\mathrm{i}}\int_{\Omega_0(0,\tau)}\frac{\mathrm{d}w}{(1+w\mathrm{e}^{-\mathrm{i}\theta})^b w^{N_0}(w-\tau)}, \tag{2.5}$$

where we choose $\Omega_0(0,\tau)$ to be the union of the circles $|w|=1-\varepsilon$ and $|w-\tau|=\frac{1}{2}\varepsilon$, where $0<\varepsilon<\frac{1}{2}$ and ε is bounded away from 0. For $|\tau|\leqslant 1-\frac{1}{2}\varepsilon$, we have to adjust this contour (see Fig. 1). Now we use the fact that $(1+w)^{-b}$ is analytic and bounded by ε^{-b} on $|w+1|\geqslant\varepsilon$. With (2.5) we obtain

$$|f_1(\tau)|\leqslant C_1(1-\varepsilon)^{-N_0}, \tag{2.6}$$

where $C_1=\varepsilon^{-b-1}4\pi(1-\frac{1}{2}\varepsilon)$. This estimate holds for $|\tau+\mathrm{e}^{\mathrm{i}\theta}|\geqslant\frac{3}{2}\varepsilon$.

We substitute (2.4) in (2.1) and obtain

$$U(a,c,z)=z^{-a}\sum_{n=0}^{N_0-1}(-1)^n\frac{(a)_n(b)_n}{n!\,z^n}+\frac{1}{\Gamma(a)}R_{N_0}(z), \tag{2.7}$$

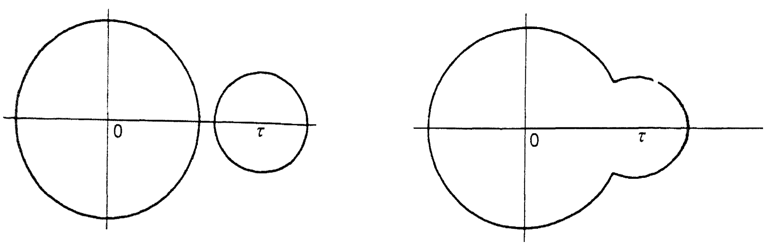

FIG. 1. The normal and adjusted contour of integration of (2.5).

with

$$R_{N_0}(z) = \mathrm{e}^{-\mathrm{i}a\theta}\int_0^\infty \mathrm{e}^{-\rho\tau}\tau^{a-1+N_0}f_1(\tau)\,\mathrm{d}\tau. \tag{2.8}$$

So with (2.6) we estimate $R_{N_0}(z)$, with

$$|R_{N_0}(z)| \leqslant C_1\int_0^\infty \mathrm{e}^{-\rho\tau}\tau^{a-1+N_0}(1-\varepsilon)^{-N_0}\,\mathrm{d}\tau. \tag{2.9}$$

At this stage we can minimize $R_{N_0}(z)$ as a function of N_0 by recognizing a gamma function on the right-hand side of (2.9), but we use a different method, which is also applicable in the following sections. The integrand has a maximum at $\tau = \gamma_1$, where γ_1 satisfies $\rho = (a-1+N_0)/\gamma_1$. This maximum is

$$\mathrm{e}^{-\rho\gamma_1}\gamma_1^{\rho\gamma_1}(1-\varepsilon)^{a-1-\rho\gamma_1},$$

and it is the main factor in asymptotic estimations of the right-hand side of (2.9). This factor as a function of γ_1 is minimal at $\gamma_1 = 1-\varepsilon$. But this is not the final choice of γ_1, since N_0 must take integer values. So we choose $N_0 = [\rho(1-\varepsilon) - a + 1]$ and afterwards we take $\gamma_1 = (a-1+N_0)/\rho$, where $[y]$ denotes the integer part of y. Thus $\gamma_1 \leqslant 1-\varepsilon$. This z dependence of N_0 is typical in the zeroth-level hyperasymptotic expansion (2.7).

Now we use Laplace's method [8: p. 80] to obtain

$$\left.\begin{aligned} |R_{N_0}(z)| &= O\big(C_1(2\pi\gamma_1/\rho)^{\frac12}\mathrm{e}^{-\rho\gamma_1}\gamma_1^{a-1+N_0}(1-\varepsilon)^{-N_0}\big)\\ &= O\big(C_1(2\pi/\rho)^{\frac12}\mathrm{e}^{-\rho\gamma_1}\gamma_1^{a-\frac12}\big)\end{aligned}\right\} \quad \text{as } |z|\to\infty. \tag{2.10}$$

Generally, a remainder $|R_n(z)|$ is of order z^{-n} as $|z|\to\infty$, but we have taken $N_0 \sim \gamma_1 z$, and we obtain $\mathrm{e}^{-|z|\gamma_1}$ as the main factor in (2.10). This is the exponential improvement of the remainder at the zeroth level.

3. First-level hyperasymptotic expansions

In this section we expand the remainder $R_{N_0}(z)$ in a new asymptotic series. Because of the optimal use of asymptotic series (2.3), this new series should be different from the remaining terms of (2.3) representing $R_{N_0}(z)$. To obtain this new series, we expand $f_1(\tau)$ at $\tau = \gamma_1$, where the main part of the integrand of (2.8), that is $\mathrm{e}^{-\rho\tau}\tau^{a-1+N_0}$, attains its maximum. So we write

$$f_1(\tau) = a_{0,1} + a_{1,1}(\tau-\gamma_1) + \cdots + a_{N_1-1,1}(\tau-\gamma_1)^{N_1-1} + (\tau-\gamma_1)^{N_1}f_2(\tau), \tag{3.1}$$

where

$$f_2(\tau) = \frac{1}{2\pi\mathrm{i}}\int_{\Omega_1(\gamma_1,\tau)} \frac{f_1(w)}{(w-\gamma_1)^{N_1}(w-\tau)}\,\mathrm{d}w, \tag{3.2}$$

and where $\Omega_1(\gamma_1, \tau)$ is a contour that encircles γ_1 and τ.

At this stage the role of the parameter ε in (2.6) is obvious. In the estimate (2.9) we need only an estimate of $|f_1(\tau)|$ on $\tau \geqslant 0$, and this would avoid the ε part in (2.9). But in (3.2) we need an estimate of $|f_1(w)|$ for w on as large as possible circle around $w = \gamma_1$.

For the optimality of the estimation of $|f_n(\tau)|$, we choose here and below for $n = 1, 2, 3, \ldots$

$$\Omega_n(\gamma_n, \tau) = \{w \in \mathbb{C} : |w - \gamma_n| = (\gamma_n^2 + 2\gamma_n \cos\theta + 1)^{\frac{1}{2}} - \rho_n\varepsilon \text{ or } |w - \tau| = 2^{-n-1}\varepsilon\}, \tag{3.3}$$

where $\rho_n = 2 - 2^{-n}$. Again, for $|\gamma_n - \tau| \leqslant (\gamma_n^2 + 2\gamma_n \cos\theta + 1)^{\frac{1}{2}} - \varepsilon\rho_{n+1}$, we have to adjust this contour. These contours look like those in Fig. 1. The square root in (3.3) is due to the singularity of f_1 at $-e^{i\theta}$.

We substitute (3.1) in (2.8) and obtain

$$R_{N_0}(z) = \Gamma(a + N_0)z^{-a} \sum_{n=0}^{N_1-1} a_{n,1}P_n^1(\gamma_1)\rho^{-(N_0+n)} + R_{N_1}(z), \tag{3.4}$$

where

$$P_n^1(\gamma_1) = \frac{\rho^{a+N_0+n}}{\Gamma(a + N_0)} \int_0^\infty e^{-\rho\tau}\tau^{a-1+N_0}(\tau - \gamma_1)^n \, d\tau = \frac{1}{\Gamma(\gamma_1\rho + 1)} \int_0^\infty e^{-x}x^{\gamma_1\rho}(x - \gamma_1\rho)^n \, dx \tag{3.5}$$

and

$$R_{N_1}(z) = e^{-ia\theta}\int_0^\infty e^{-\rho\tau}\tau^{a-1+N_0}(\tau - \gamma_1)^{N_1}f_2(\tau) \, d\tau. \tag{3.6}$$

The functions $P_n^1(\gamma_1)$ are polynomials in $\gamma_1\rho$ and will be discussed in Section 5.

We estimate $f_2(\tau)$ on $|\tau + e^{i\theta}| \geqslant \rho_2\varepsilon$ as follows:

$$|f_2(\tau)| \leqslant C_2(1 - \varepsilon)^{-N_0}[(\gamma_1^2 + 2\gamma_1 \cos\theta + 1)^{\frac{1}{2}} - \rho_1\varepsilon]^{-N_1}, \tag{3.7}$$

where $C_2 = 2\pi[(\gamma_1^2 + 2\gamma_1 \cos\theta + 1)^{\frac{1}{2}} - (\rho_1 - \frac{1}{4})\varepsilon](4/\varepsilon)C_1$. This gives with (3.6)

$$|R_{N_1}(z)| \leqslant \frac{C_2}{(1 - \varepsilon)^{N_0}} \int_0^\infty e^{-\rho\tau}\tau^{a-1+N_0} |\tau - \gamma_1|^{N_1}[(\gamma_1^2 + 2\gamma_1 \cos\theta + 1)^{\frac{1}{2}} - \rho_1\varepsilon]^{-N_1} \, d\tau. \tag{3.8}$$

Again, to obtain the optimal N_1, we minimize the right-hand side of (3.8). In this case the integrand is of the form shown in Fig. 2. The local maxima are at $\tau = s_j$ $(j = 1, 2)$, and these s_j satisfy

$$\rho = \frac{a - 1 + N_0}{s_j} + \frac{N_1}{s_j - \gamma_1}. \tag{3.9}$$

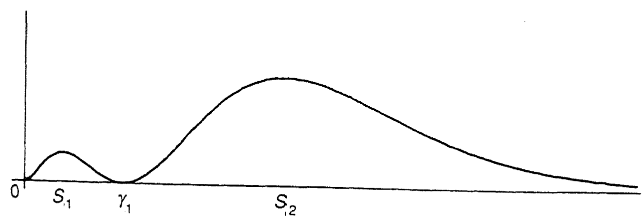

FIG. 2. The graph of the integrand of (3.8).

Further analysis shows that for all $N_1 > 0$ the global maximum is attained at s_2, and this maximum is minimal at $s_2 - \gamma_1 = (\gamma_1^2 + 2\gamma_1 \cos\theta + 1)^{\frac{1}{2}} - \rho_1 \varepsilon$. Now N_1 follows from (3.9), giving $N_1 = \rho[(\gamma_1^2 + 2\gamma_1\cos\theta + 1)^{\frac{1}{2}} - \rho_1\varepsilon]^2[\gamma_1 + (\gamma_1^2 + 2\gamma_1\cos\theta + 1)^{\frac{1}{2}} - \rho_1\varepsilon]^{-1}$. So we choose

$$N_1 = \left[\rho[(\gamma_1^2 + 2\gamma_1\cos\theta + 1)^{\frac{1}{2}} - \rho_1\varepsilon]^2[\gamma_1 + (\gamma_1^2 + 2\gamma_1\cos\theta + 1)^{\frac{1}{2}} - \rho_1\varepsilon]^{-1}\right], \quad (3.10)$$

and $\gamma_2 = s_2$, that is the largest solution of (3.9). It is not difficult to show that s_2 is an increasing function of N_1, so we have $\gamma_2 \leq \gamma_1 + (\gamma_1^2 + 2\gamma_1\cos\theta + 1)^{\frac{1}{2}} - \rho_1\varepsilon$. And we obtain

$$\left.\begin{aligned} |R_{N_1}(z)| &= O\left(C_2(2\pi\gamma_2/\rho)^{\frac{1}{2}} e^{-\rho\gamma_2} \frac{\gamma_2^{a-1+N_0}}{(1-\varepsilon)^{N_0}} \left(\frac{\gamma_2 - \gamma_1}{(\gamma_1^2 + 2\gamma_1\cos\theta + 1)^{\frac{1}{2}} - \rho_1\varepsilon}\right)^{N_1}\right) \\ &= O\left(C_2(2\pi\gamma_2/\rho)^{\frac{1}{2}} \gamma_1^{a-1} e^{-\rho\gamma_2} (\gamma_2/\gamma_1)^{\rho\gamma_1}\right) \end{aligned}\right\} \quad \text{as } |z| \to \infty. \quad (3.11)$$

From the previous analysis it follows that $\gamma_1 = 1 - \varepsilon + O(\rho^{-1})$ and $\gamma_2 = [2(1-\varepsilon)(1 + \cos\theta) + \varepsilon^2]^{\frac{1}{2}} + 1 - \frac{5}{2}\varepsilon + O(\rho^{-1})$. Substituting these results in (3.11) and taking ε small and $\theta = 0$, we obtain the estimate

$$|R_{N_1}(z)| = O(z^{-\frac{1}{2}} e^{-(1.90\cdots)z}) \quad \text{as } z \to \infty. \quad (3.12)$$

From (2.10) it follows that for small ε we have the estimate $|R_{N_0}(z)| = O(z^{-\frac{1}{2}}e^{-z})$. So $R_{N_1}(z)$ is exponentially small compared with $R_{N_0}(z)$.

Remark 1. So far, we have fixed z and N_0 and then calculated N_1. Now we take $\theta = 0$. For the zeroth-level hyperasymptotics it is optimal to take $\gamma_1 = (a - 1 + N_0)/z \approx 1 - \varepsilon$. But this is not the optimal value for first-level hyperasymptotics. In the calculations of N_1 we did not use $\gamma_1 \approx 1 - \varepsilon$, so given $\gamma_1 > 0$ the preceding analysis shows that the optimal $\gamma_2 = s_2$ is $\gamma_2 = 1 + 2\gamma_1 - \frac{3}{2}\varepsilon$. Thus the first-order estimate (3.11) can be regarded as a function of z, ε, and γ_1. Keeping z and ε fixed, it is easy to show that the optimal γ_1 satisfies

$$\ln\frac{1 + 2\gamma_1 - \frac{3}{2}\varepsilon}{1 - \varepsilon} = \frac{2 + 2\gamma_1 - 3\varepsilon}{1 + 2\gamma_1 - \frac{3}{2}\varepsilon}.$$

Thus

$$\gamma_1 = \alpha + \left(\frac{1}{8(\alpha + 1)} - \alpha\right)\varepsilon + O(\varepsilon^2) \quad \text{as } \varepsilon \downarrow 0,$$

with $\alpha = 1.29556\ldots$. Now we have to adjust this γ_1 such that $N_0 = z\gamma_1 - a + 1$ is an integer. We obtain $\gamma_1 = \alpha + c\varepsilon + O(z^{-1})$ and $\gamma_2 = 1 + 2\alpha + d\varepsilon + O(z^{-1})$ as $z \to \infty$, where c and d are of the form constant $+ O(\varepsilon)$ as $\varepsilon \downarrow 0$. Substituting these results in the first estimate of (3.11), we obtain for small ε the estimate $|R_{N_1}(z)| = O(z^{-\frac{1}{2}}e^{-(1.93\ldots)z})$, which is even better than (3.12).

Notice that it is possible to calculate the optimal N_0 and N_1 because of the small number of free variables. In order to obtain optimal nth-level hyperasymptotic expansions, we have to find the optimal $N_0, N_1, \ldots, N_n$. Generally, this is too difficult; consequently in higher-level hyperasymptotics we assume z, $N_0, \ldots, N_{n-1}$ fixed and then calculate N_n.

Remark 2. The parameter θ has a large influence on the exponential improvements. It follows from (3.10) that as $\theta\uparrow\pi$ then $N_1\downarrow 0$; in consequence the exponential improvement of the first level diminishes.

4. Higher-level hyperasymptotic expansions

In this section we use the method of Section 3 for obtaining a second-level hyperasymptotic expansion. The main part of the integrand of (3.6) is $e^{-\rho\tau}\tau^{a-1+N_0}|\tau-\gamma_1|^{N_1}$, and this function is maximal at $\tau=\gamma_2$. Thus we expand $f_2(\tau)$ at this point:

$$f_2(\tau)=a_{0,2}+a_{1,2}(\tau-\gamma_2)+\cdots+a_{N_2-1,2}(\tau-\gamma_2)^{N_2-1}+(\tau-\gamma_2)^{N_2}f_3(\tau), \quad (4.1)$$

where

$$f_3(\tau)=\frac{1}{2\pi i}\int_{\Omega_2(\gamma_2,\tau)}\frac{f_2(w)}{(w-\gamma_2)^{N_2}(w-\tau)}\,dw. \quad (4.2)$$

We substitute (4.1) in (3.6) and obtain

$$R_{N_1}(z)=\Gamma(a+N_0)z^{-a}\sum_{m=0}^{N_2-1}a_{m,2}P^2_{N_1,m}(\gamma_1,\gamma_2)\rho^{-(N_0+N_1+m)}+R_{N_2}(z), \quad (4.3)$$

where

$$P^2_{n,m}(\gamma_1,\gamma_2)=\frac{\rho^{a+N_0+n+m}}{\Gamma(a+N_0)}\int_0^\infty e^{-\rho\tau}\tau^{a-1+N_0}(\tau-\gamma_1)^n(\tau-\gamma_2)^m\,d\tau, \quad (4.4)$$

and where

$$R_{N_2}(z)=e^{-ia\theta}\int_0^\infty e^{-\rho\tau}\tau^{a-1+N_0}(\tau-\gamma_1)^{N_1}(\tau-\gamma_2)^{N_2}f_3(\tau)\,d\tau. \quad (4.5)$$

Now we estimate $f_3(\tau)$ on $|\tau+e^{i\theta}|\geq\rho_3\varepsilon$ with

$$|f_3(t)|\leq C_3(1-\varepsilon)^{-N_0}[(\gamma_1^2+2\gamma_1\cos\theta+1)^{\frac{1}{2}}-\rho_1\varepsilon]^{-N_1}$$
$$\times[(\gamma_2^2+2\gamma_2\cos\theta+1)^{\frac{1}{2}}-\rho_2\varepsilon]^{-N_2}, \quad (4.6)$$

where $C_3=2\pi[(\gamma_2^2+2\gamma_2\cos\theta+1)^{\frac{1}{2}}-(\rho_2-\frac{1}{8})\varepsilon](8/\varepsilon)C_2$. With (4.5) this gives

$$|R_{N_2}(z)|\leq C_3'\int_0^\infty e^{-\rho\tau}\tau^{a-1+N_0}|\tau-\gamma_1|^{N_1}\left(\frac{|\tau-\gamma_2|}{(\gamma_2^2+2\gamma_2\cos\theta+1)^{\frac{1}{2}}-\rho_2\varepsilon}\right)^{N_2}d\tau, \quad (4.7)$$

where $C_3'=C_3(1-\varepsilon)^{-N_0}[(\gamma_1^2+2\gamma_1\cos\theta+1)^{\frac{1}{2}}-\rho_1\varepsilon]^{-N_1}$. Again, we want to minimize the right-hand side to obtain N_2. Now the integrand is of the form shown in Fig. 3.

The local maxima are at $\tau=s_j$ $(j=1,2,3)$, and these s_j satisfy

$$\rho=\frac{a-1+N_0}{s_j}+\frac{N_1}{s_j-\gamma_1}+\frac{N_2}{s_j-\gamma_2}, \quad (4.8)$$

and γ_1, N_0, and N_1 are known from the earlier sections. Thus s_1, s_2, and s_3 are functions of N_2. The most interesting maximum is at s_3, because this maximum,

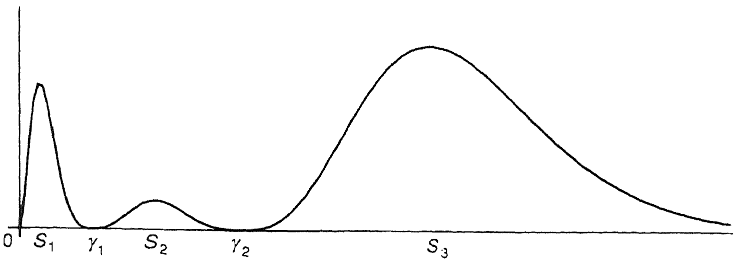

FIG. 3. The graph of the integrand of (4.7).

as a function of $s_3(N_2)$, is minimal if $s_3 - \gamma_2 = (\gamma_2^2 + 2\gamma_2 \cos\theta + 1)^{\frac{1}{2}} - \rho_2\varepsilon$. The other two maxima are decreasing functions of N_2. For some small N_2 it is possible that the global maximum is at s_1, but for the special $s_3(\tilde{N}_2) = \gamma_2 + (\gamma_2^2 + 2\gamma_2 \cos\theta + 1)^{\frac{1}{2}} - \rho_2\varepsilon$ the parameter $\tilde{N}_2$ is not small, and it is conjectured that for $N_2 \geqslant \tilde{N}_2$ the global maximum is at s_3.

Nevertheless, for some $N_2 \in \mathbb{N}$ the global maximum at s_j is minimal, and we can take $\gamma_3 = s_j$. Substituting these results in the right-hand side of (4.7) gives an order estimate of $|R_{N_2}|$, which is exponentially small compared with the order estimate of $|R_{N_1}|$; compare (3.11). Furthermore, we can proceed with this method and obtain nth-level hyperasymptotic expansions. Then the remainders can be estimated by

$$|R_{N_n}(z)| = O(\rho^{-\frac{1}{2}} e^{-\lambda_n \rho}) \quad \text{as } |z| \to \infty, \tag{4.9}$$

where $\lambda_0 \sim 1 < \lambda_1 < \lambda_2 < \lambda_3 < \cdots$. But it is not clear that $\{\lambda_n\}_{n\in\mathbb{N}}$ is unbounded. For $\theta = 0$ we have $\lambda_1 \sim 1.90$, and if the conjecture holds, and the optimal global maximum is at $\gamma_3 \approx 1 + 2\gamma_2 - \rho_2\varepsilon$, then we obtain the result $\lambda_2 \sim 3.59$. Thus, at the second level, our λ_2 would already exceed Berry and Howls' limiting value of 2.39.

Remark 1. The new method presented here is not restricted to the confluent hypergeometric U-function. Suppose that $F(z)$ has an integral representation of the form

$$F(z) = \int_0^\infty t^{a-1} e^{-zt} f(t)\, dt, \tag{4.10}$$

where $a > 0$, and where $f(t)$ is an analytic function on a sector containing $[0, \infty)$, and

$$|f(t)| \leqslant e^{\sigma t}(C + Dt^M), \qquad t \in [0, \infty), \tag{4.11}$$

with $\sigma \in \mathbb{R}$, $C, D \in \mathbb{R}^+$, and $M \in \mathbb{N}$. Then, by changing z in $z - \sigma$ and adjusting the estimations of $f_n(t)$, the same method can be used. But the calculations of the N_n will be more difficult, of course.

Remark 2. The method described in this paper can be extended to complex a and b. By analytic continuation with respect to a, we see that (3.6) remains valid for $\operatorname{Re} a > -N_0$. Thus our method can be extended to this half-plane by the replacement of a by Re a at several places. The extension to complex b can be obtained by taking a slightly different C_1 in the estimate (2.6).

5. Recursion relations for the polynomial coefficients

In Sections 3 and 4 the following polynomials arise:

$$\left.\begin{aligned} P_n^1(\gamma_1) &= \frac{1}{\Gamma(\gamma_1\rho+1)}\int_0^\infty e^{-x}x^{\gamma_1\rho}(x-\gamma_1\rho)^n\,dx, \\ P_{n,m}^2(\gamma_1,\gamma_2) &= \frac{1}{\Gamma(\gamma_1\rho+1)}\int_0^\infty e^{-x}x^{\gamma_1\rho}(x-\gamma_1\rho)^n(x-\gamma_2\rho)^m\,dx, \end{aligned}\right\} \tag{5.1}$$

where $P_{n,m}^2(\gamma_1,\gamma_2)$ can be seen as the second-level generalization of $P_n^1(\gamma_1)$, also used by Temme [11]. For calculations of expansions in terms of the $P_n^1(\gamma_1)$, the following recursion relation is useful:

$$P_{n+1}^1(\gamma_1) = (n+1)P_n^1(\gamma_1) + \gamma_1\rho n P_{n-1}^1(\gamma_1), \qquad P_0^1(\gamma_1) = P_1^1(\gamma_1) = 1. \tag{5.2}$$

Now it easily follows that $P_n^1(\gamma_1)$ is a polynomial in $\gamma_1\rho$ with integer coefficients and of order $[\frac{1}{2}n]$.

The expansions in Section 4 are in terms of $P_{n,m}^2(\gamma_1,\gamma_2)$, where $n = N_1$ is fixed. So we are only interested in a recursion relation with respect to the m-index. This more complicated recursion relation is given by

$$\begin{aligned} P_{n,m+1}^2(\gamma_1,\gamma_2) = {} & [n+m+1+2(\gamma_1-\gamma_2)\rho]P_{n,m}^2 \\ & + \{\gamma_2\rho(n+m) - [m+(\gamma_1-\gamma_2)\rho](\gamma_1-\gamma_2)\rho\}P_{n,m-1}^2 \\ & - (m-1)\gamma_2(\gamma_1-\gamma_2)\rho^2 P_{n,m-2}^2. \end{aligned} \tag{5.3}$$

The proof of (5.3) uses some simple relations in terms of the P_j^1 and $P_{n,m}^2$, and is omitted. By expression $(x-\gamma_2\rho)^m$ in terms of $(x-\gamma_1\rho)^j$, we obtain

$$P_{n,m}^2 = \sum_{j=0}^m \binom{m}{j}[(\gamma_1-\gamma_2)\rho]^j P_{n+m-j}^1, \tag{5.4}$$

which can be used for obtaining simple expressions of $P_{n,0}^2$, $P_{n,1}^2$, and $P_{n,2}^2$ in terms of the (known) P_j^1, which are needed as initial terms of recursion relation (5.3).

In higher-level hyperasymptotics, the following generalizations of (5.1) arise:

$$P_{n_1,\ldots,n_k}^k(\gamma_1,\ldots,\gamma_k) = \frac{1}{\Gamma(\gamma_1\rho+1)}\int_0^\infty e^{-x}x^{\gamma_1\rho}(x-\gamma_1\rho)^{n_1}\cdots(x-\gamma_k\rho)^{n_k}\,dx. \tag{5.5}$$

These k-level polynomials also have (complicated) recursion relations, and are expressible in terms of the $(k-1)$-level polynomials, just as (5.4).

6. Numerical illustrations

The first illustration is the K-Bessel function. We choose $a = b = \frac{1}{2}$ and have

$$U(\tfrac{1}{2}, 1, z) = \pi^{-\frac{1}{2}}e^{\frac{1}{2}z}K_0(\tfrac{1}{2}z). \tag{6.1}$$

TABLE 1
Hyperasymptotic approximations to $U(\frac{1}{2}, 1, z)$ for $z = 10$

Level	Approximation	|Approx. − Exact|	ε	
Zeroth	0.30906549905615001306	1.823×10^{-6}		$N_0 = 10$
First	0.30906732141838357662	1.560×10^{-10}	$\frac{1}{10}$	$N_1 = 12$
Second	0.30906732157435500491	9.051×10^{-18}		$N_2 = 23$
Zeroth	0.30906906160073788173	1.740×10^{-6}		$N_0 = 11$
First	0.30906732168508116748	1.107×10^{-10}	0	$N_1 = 14$
Second	0.30906732157435499781	1.951×10^{-18}		$N_2 = 26$
Exact	0.30906732157435499585	0		

We use the method described in the previous sections with $z = 10$. In this case it is provable that in the second-level contribution the global maximum of the integrand of (4.7) is minimal when $s_3 - \gamma_2 = 1 + \gamma_2 - (2\varepsilon - \frac{1}{4}\varepsilon)$, and this maximum is attained at s_3 such that the number of second-level terms, N_2, is easy to compute. The results are given in Table 1.The improvements are obvious. Although the case $\varepsilon = 0$ is excluded in the previous sections, the approximation formulae still hold, and these approximations are even better than in the $\varepsilon = \frac{1}{10}$ case. It follows that in some cases the estimates of $|f_n|$, for $\varepsilon \downarrow 0$, are too rough, and in these cases the accompanying C_n can be replaced by C_n', which do not depend on ε.

All numerical work was performed on a Sun-4/280 using the program Maple V (see [5]), which has the advantages that it can be configured to work to any specified accuracy and that it can compute the difficult coefficients of the Taylor series which we use in (2.2), (3.1), and (4.1). A consequence of the existence of these formula-manipulating packages is that our method is easy to program for general a, b, z, and ε.

The second numerical illustration is the same illustration given in [2]. There, the hyperasymptotic expansion of the function

$$Y(z) = 2\pi^{\frac{1}{2}}(\tfrac{3}{4}z)^{\frac{1}{6}}e^{\frac{1}{2}z}\mathrm{Ai}[(\tfrac{3}{4}z)^{\frac{2}{3}}] \tag{6.2}$$

is obtained with the second-order differential equation of which $Y(z)$ is a solution. As mentioned in the introduction, this expansion is finite for fixed z. The final result is an approximation at $z = 16$ with an error 1.151×10^{-18}. In order to obtain our hyperasymptotic approximation of the function $Y(z)$, we choose $a = \frac{5}{6}$ and $b = \frac{1}{6}$. Then we have $Y(z) = z^{\frac{5}{6}}U(\frac{5}{6}, \frac{5}{3}, z)$. The results of our hyperasymptotic approximation up to the second level are given in Table 2. At the second level our approximation is better than the one obtained from the differential equation. To obtain this result we needed 70 terms. This is more than twice the optimal number of terms in the final 'differential-equation approximation'. Our third-level approximation differs by only 2.873×10^{-34} from the exact value, and we needed $N_3 = 71$ new terms. It would take 35 digits to show the difference in the table, so we omit this result. Notice that, in contrast to the N_j in [2], our first four N_j are increasing.

TABLE 2
Hyperasymptotic approximations to Y for $z = 16$ and $\varepsilon = \frac{1}{10}$

Level	Approximation	|Approx. − Exact|	
Zeroth	0.991836805434937558499124768	6.247×10^{-9}	$N_0 = 15$
First	0.991836799188260218215670711	2.381×10^{-15}	$N_1 = 19$
Second	0.991836799188262598909796631	1.117×10^{-26}	$N_2 = 36$
Exact	0.991836799188262598909796642	0	

As $\theta \uparrow \pi$, the exponential improvements at the higher levels diminish. This can be illustrated by taking $z = 16\mathrm{i}$ in the previous numerical illustration. The results are given in Table 3. Notice that the results in Table 2 are better, and that in this illustration the first three N_j are not increasing.

In the more theoretical part of this paper we calculated the cut-off places by minimization of the remainder. To give an idea of what happens with the various terms in our approximations, we present Fig. 4, which shows the decrease of the absolute value of the terms in levels 0, 1, and 2 in the hyperasymptotic approximation of $Y(z)$, for $z = 16$. It seems that the cut-off places coincide with the smallest terms of the corresponding asymptotic expansions at each level. But even at the first level it is much harder to obtain the optimal N_1 by calculating the smallest term than by minimizing $|R_{N_1}(z)|$.

7. Comparison with other papers

Our hyperasymptotic expansion of level greater than 0 is limited to the sector $|\theta| < \pi$. This is due to new exponentially small terms in the asymptotic expansion of $U(a, c, z)$, which appear in neighbourhoods of the so-called Stokes lines $\theta = \pm\pi$. These new terms are of the same order as $R_{N_0}(z)$, and they are not covered by our hyperasymptotic expansion.

Recently, in [9], a new uniform expansion for $U(a, c, z)$ has been obtained that has as θ-sector given by $|\theta| < \frac{5}{2}\pi$. In that paper Olver uses the representation

$$R_{N_0}(z) = (-1)^{N_0} \frac{\sin \pi b}{\pi} \int_0^\infty \frac{\mathrm{e}^{-z\tau}\tau^{a-1+N_0}}{1+\tau}\, \mathrm{d}\tau \int_0^\infty \mathrm{e}^{-z\tau v} v^{-b}(1+v)^{a-1}\, \mathrm{d}v. \tag{7.1}$$

He substitutes the expansion

$$(1+v)^{a-1} = \sum_{s=0}^{N_1-1} (-1)^s \frac{(1-a)_s}{s!} v^s + (1-a)_{N_1} v^{N_1} \phi_{N_1}(a, v) \tag{7.2}$$

TABLE 3
Hyperasymptotic approximations to Y for $z = 16\mathrm{i}$ and $\varepsilon = \frac{1}{10}$

Level	|Approx. − Exact|	
Zeroth	8.678×10^{-9}	$N_0 = 15$
First	1.191×10^{-11}	$N_1 = 11$
Second	3.412×10^{-18}	$N_2 = 20$

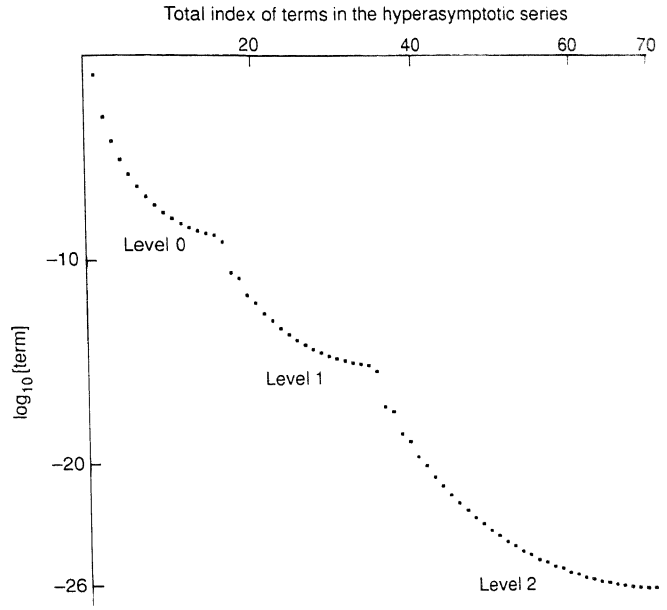

FIG. 4. Decrease of the first three hyperseries of $Y(16)$.

in (7.1), and obtains

$$R_{N_0}(z) = (-1)^{N_0} 2\pi \frac{z^{b-1}e^z}{\Gamma(b)} \sum_{s=0}^{N_1-1} (-1)^s \frac{(1-a)_s(1-b)_s F_{N_0-s+a+b-1}(z)}{s!\, z^s} + R_{N_1}(z), \quad (7.3)$$

where

$$F_m(z) = \frac{e^{-z}}{2\pi} \int_0^\infty \frac{e^{-zt} t^{m-1}}{1+t}\, dt \quad (7.4)$$

and

$$R_{N_1}(z) = (-1)^{N_0}(1-a)_{N_1} \frac{\sin \pi b}{\pi} \int_0^\infty \frac{e^{-z\tau}\tau^{a-1+N_0}}{1+\tau}\, d\tau \int_0^\infty e^{-z\tau v} v^{N_1-b} \phi_{N_1}(a, v)\, dv. \quad (7.5)$$

The problem in obtaining an estimate for $R_{N_1}(z)$ in the sector $|\theta| < \frac{5}{2}\pi$ is the singularity in the τ integrand at $\tau = -1$. With

$$\frac{e^{-z\tau v}}{1+\tau} = \frac{e^{zv}}{1+\tau} + \frac{e^{-z\tau v} - e^{zv}}{1+\tau}, \quad (7.6)$$

the remainder $R_{N_1}(z)$ is split up into a sum of two double integrals, which are easier to handle. The final result is

$$\left.\begin{aligned} R_{N_1}(z) &= O(e^{-|z|} z^{b-N_1-1}) \quad \text{for } |\theta| \leq \pi, \\ R_{N_1}(z) &= O(e^{z} z^{b-N_1-1}) \quad \text{for } \pi \leq |\theta| \leq \tfrac{5}{2}\pi - \delta, \end{aligned}\right\} \quad (7.7)$$

as $|z| \to \infty$.

Olver's expansion is a first-level expansion in the sector $|\theta| \leq \pi$, but in the larger sector the exponential improvements of the first level gradually diminish.

Even in the sector $|\theta| \leq \pi$ it is not easy to obtain higher-level expansions from Olver's first-level expansion.

In a recent paper of Boyd [4], a method is given for obtaining first level expansions. The method is based on Stieltjes transforms, and it is indicated how higher-level expansions can be obtained. The method uses the representation

$$U(a, c, z) = \frac{z^{1-a}}{\Gamma(a)\Gamma(b)} \int_0^\infty \frac{U(-a+c, c, t)e^{-t}t^{a-1}}{t+z} \, dt, \tag{7.8}$$

and is elaborated for the modified Bessel function $K_\nu(z)$. The integral representations for the remainders R_{N_l} are the same as our integral representations, with $f_l(t)$ replaced by $U(-a+c, c, t)/(1+t)$.

In the case of the $K_\nu(z)$, N_1 is obtained by tracing the smallest term in the first-level expansion. At this first level Boyd obtains for $\theta = 0$ an exponential improvement which is of the same order as (3.12).

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References

1. Airey, J. R. 1937 The 'converging factor' in asymptotic series and the calculation of Bessel, Laguerre and other functions. *Phil. Mag.* **24,** 521–52.
2. Berry, M. V., & Howls, C. J. 1990 Hyperasymptotics. *Proc. R. Soc. Lond.* A **430,** 653–67.
3. Berry, M. V., & Howls, C. J. 1991 Hyperasymptotics for integrals with saddles. *Proc. R. Soc. Lond.* A **434,** 657–75.
4. Boyd, W. G. C. 1990 Stieltjes transforms and the Stokes phenomenon. *Proc. R. Soc. Lond.* A **429,** 227–46.
5. Char, B. W., *et al.* 1991 *Maple Library Reference Manual.* New York: Springer.
6. Dingle, R. B. 1973 *Asymptotic Expansions: Their Derivation and Interpretation.* London: Academic Press.
7. Miller, J. C. P. 1952 A method for the determination of converging factors, applied to the asymptotic expansions for the parabolic cylinder functions. *Proc. Camb. Phil. Soc.* **48,** 243–54.
8. Olver, F. W. J. 1974 *Asymptotics and Special Functions.* London: Academic Press.
9. Olver, F. W. J. 1991 Uniform, exponentially improved, asymptotic expansions for the confluent hypergeometric function and other integral transforms. *SIAM J. Math. Anal.* **22,** 1475–89.
10. Stieltjes, T. J. 1886 Recherches sur quelques séries semi-convergentes. *Ann. Sci. École Norm. Sup.* **3,** 201–58. Reprinted in *Complete Works,* Vol. 2, pp. 2–58. Groningen: Noordhoff (1918).
11. Temme, N. M. 1983 Uniform asymptotic expansions of Laplace integrals. *Analysis* **3,** 221–49.